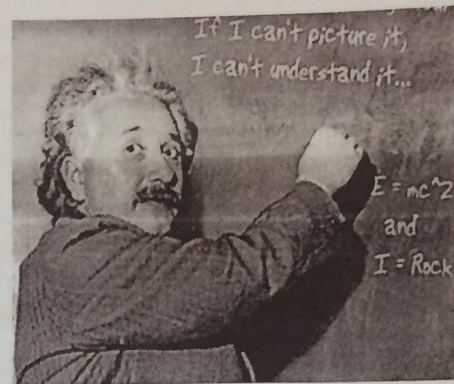
3.4 Radical Ideas

A Practice Understanding Task



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Now that Tia and Tehani know that $a^{m/n} = \left(\sqrt[n]{a}\right)^m$ they are wondering which form, radical form or exponential form, is best to use when working with numerical and algebraic expressions.

Tia says she prefers radicals since she understands the following properties for radicals (and there are not too many properties to remember):

If n is a positive integer greater than 1 and both a and b are positive real numbers then,

1. $\sqrt[n]{a^n} = a$
2. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
3. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Tehania says she prefers exponents since she understands the following properties for exponents (and there are more properities to work with):

1. $a^m \cdot a^n = a^{m+n}$
2. $\left(a^m\right)^n = a^{mn}$
3. $(ab)^n = a^n \cdot b^n$
4. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
5. $\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$
6. $a^{-n} = \dfrac{1}{a^n}$

$a^{\frac{1}{n}} \cdot a^{\frac{1}{n}}$ (n times) $= a^{\frac{n}{n}} = a^1$

$\sqrt[n]{a} \cdot \sqrt[n]{a}$ (n times) $= \sqrt[n]{a^n} = a^1$

Therefore, $a^{\frac{1}{n}} = \sqrt[n]{a}$

DO THIS: Illustrate with examples and explain, using the properties of radicals and exponents, why $a^{1/n} = \sqrt[n]{a}$ and $a^{m/n} = \left(\sqrt[n]{a}\right)^m$ are true identities.

Using their preferred notation, Tia might simplify $\sqrt[3]{x^8}$ as follows:

$$\sqrt[3]{x^8} = \sqrt[3]{x^3 \cdot x^3 \cdot x^2} = \sqrt[3]{x^3} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{x^2} = x \cdot x \cdot \sqrt[3]{x^2} = x^2 \cdot \sqrt[3]{x^2}$$

(Tehani points out that Tia also used some exponent rules in her work.)

On the other hand, Tehani might simplify $\sqrt[3]{x^8}$ as follows:

$$\sqrt[3]{x^8} = x^{8/3} = x^{2+2/3} = x^2 \cdot x^{2/3} \text{ or } x^2 \cdot \sqrt[3]{x^2}$$

For each of the following problems, simplify the expression in the ways you think Tia and Tehani might do it.

Original expression	What Tia and Tehani might do to simplify the expression:
$\sqrt{27}$	Tia's method $\sqrt{27} = \sqrt{3\cdot 3\cdot 3} = \sqrt{3^2}\cdot\sqrt{3} = 3\sqrt{3}$
	Tehani's method $27^{\frac{1}{2}} = (9\cdot 3)^{\frac{1}{2}} = 9^{\frac{1}{2}}\cdot 3^{\frac{1}{2}} = 3\cdot 3^{\frac{1}{2}}$ or $3\sqrt{3}$
$\sqrt[3]{32}$	Tia's method $\sqrt[3]{32} = \sqrt[3]{2^3\cdot 2^2} = \sqrt[3]{2^3}\cdot\sqrt[3]{2^2} = 2\cdot\sqrt[3]{2^2} = 2\cdot\sqrt[3]{4}$
	Tehani's method $32^{\frac{1}{3}} = (8\cdot 4)^{\frac{1}{3}} = 8^{\frac{1}{3}}\cdot 4^{\frac{1}{3}} = 2\cdot 4^{\frac{1}{3}}$ or $2\sqrt[3]{4}$
$\sqrt{20x^7}$	Tia's method $\sqrt{4\cdot 5\cdot x^2\cdot x^2\cdot x^2\cdot x} = \sqrt{4}\cdot\sqrt{5}\cdot\sqrt{x^2}\cdot\sqrt{x^2}\cdot\sqrt{x^2}\cdot\sqrt{x} = 2\cdot\sqrt{5}\cdot x\cdot x\cdot x\cdot\sqrt{x} = 2x^3\sqrt{5x}$
	Tehani's method $(20x^7)^{\frac{1}{2}} = (4\cdot 5\cdot x^2\cdot x^2\cdot x^2\cdot x)^{\frac{1}{2}} = 4^{\frac{1}{2}}\cdot 5^{\frac{1}{2}}\cdot (x^2)^{\frac{1}{2}}\cdot (x^2)^{\frac{1}{2}}\cdot (x^2)^{\frac{1}{2}}\cdot x^{\frac{1}{2}} = 2\cdot 5^{\frac{1}{2}}\cdot x\cdot x\cdot x\cdot x^{\frac{1}{2}} = 2x^3(5x)^{\frac{1}{2}}$
$\sqrt[3]{\frac{16xy^5}{x^7y^2}}$	Tia's method $\sqrt[3]{\frac{16xy^5}{x^7y^2}} = \frac{\sqrt[3]{16xy^5}}{\sqrt[3]{x^7y^2}} = \frac{\sqrt[3]{8\cdot 2}\cdot\sqrt[3]{xy^3\cdot y^2}}{\sqrt[3]{x^3\cdot x^3\cdot x\cdot y^2}} = \frac{2\cdot\sqrt[3]{2}\cdot\sqrt[3]{x}\cdot y\cdot\sqrt[3]{y^2}}{x\cdot x\cdot\sqrt[3]{x}\cdot\sqrt[3]{y^2}} = \frac{2y\sqrt[3]{2xy^2}}{x^2\sqrt[3]{xy^2}}$
	Tehani's method $\left(\frac{16xy^5}{x^7y^2}\right)^{\frac{1}{3}} = \frac{16^{\frac{1}{3}}x^{\frac{1}{3}}(y^5)^{\frac{1}{3}}}{(x^7)^{\frac{1}{3}}(y^2)^{\frac{1}{3}}} = \frac{\cancel{(2^3)^{\frac{1}{3}}}\cdot 2^{\frac{1}{3}}\cdot x^{\frac{1}{3}}\cdot\cancel{(y^3)^{\frac{1}{3}}}\cdot (y^2)^{\frac{1}{3}}}{\cancel{(x^3)^{\frac{1}{3}}}\cdot\cancel{(x^3)^{\frac{1}{3}}}\cdot x^{\frac{1}{3}}(y^2)^{\frac{1}{3}}} = \frac{2y(2xy^2)^{\frac{1}{3}}}{x\cdot x(xy^2)^{\frac{1}{3}}} = \frac{2y(2xy^2)^{\frac{1}{3}}}{x^2(xy^2)^{\frac{1}{3}}}$

Tia and Tehani continue to use their preferred notation when solving equations.

For example, Tia might solve the equation $(x+4)^3 = 27$ as follows:

$$(x+4)^3 = 27$$
$$\sqrt[3]{(x+4)^3} = \sqrt[3]{27} = \sqrt[3]{3^3}$$
$$x+4=3$$
$$x=-1$$

Tehani might solve the same equation as follows:

$$(x+4)^3 = 27$$
$$\left[(x+4)^3\right]^{1/3} = 27^{1/3} = (3^3)^{1/3}$$
$$x+4=3$$
$$x=-1$$

For each of the following problems, simplify the expression in the ways you think Tia and Tehani might do it.

Original equation	What Tia and Tehani might do to solve the equation:
$(x-2)^2 = 50$	Tia's method $\sqrt[2]{(x-2)^2} = \sqrt[2]{50} = \sqrt[2]{5^2 \cdot 2}$ $x-2 = 5\sqrt[2]{2} \longrightarrow x = 5\sqrt[2]{2}+2$
	Tehani's method $((x-2)^2)^{\frac{1}{2}} = (50)^{\frac{1}{2}} = (5^2 \cdot 2)^{\frac{1}{2}}$ $x-2 = 5 \cdot 2^{\frac{1}{2}} \longrightarrow x = 5 \cdot 2^{\frac{1}{2}} + 2$
$9(x-3)^2 = 4$	Tia's method $\sqrt[2]{9(x-3)^2} = \sqrt[2]{4}$ $\sqrt[2]{9} \cdot \sqrt[2]{(x-3)^2} = 2$ $3(x-3) = 2$ $\frac{3(x-3)}{3} = \frac{2}{3}$ $(x-3) = \frac{2}{3}$ $x = \frac{2}{3}+3$ $x = 3\frac{2}{3} = \frac{11}{3}$
	Tehani's method $(9(x-3)^2)^{\frac{1}{2}} = 4^{\frac{1}{2}}$ $9^{\frac{1}{2}}((x-3)^2)^{\frac{1}{2}} = (2^2)^{\frac{1}{2}}$ $3(x-3) = 2$ $\frac{3(x-3)}{3} = \frac{2}{3}$ $(x-3) = \frac{2}{3}$ $x = \frac{2}{3}+3$ $x = 3\frac{2}{3} = \frac{11}{3}$

Plug in to calculator $y_1 = (x-2)^2$ $y_2 = 50$ and $y_1 = 9(x-3)^2$ $y_2 = 4$

Zac is showing off his new graphing calculator to Tia and Tehani. He is particularly excited about how his calculator will help him visualize the solutions to equations.

"Look," Zac says. "I treat the equation like a system of two equations. I set the expression on the left equal to y_1 and the expression of the right equal to y_2, and I know at the x value where the graphs intersect the expressions are equal to each other."

Zac shows off his new method on both of the equations Tia and Tehani solved using the properties of radicals and exponents. To everyone's surprise, both equations have a second solution.

1. Use Zac's graphical method to show that both of these equations have two solutions. Determine the exact values of the solutions you find on the calculator that Tia and Tehani did not find using their algebraic methods.

$x = -5.07$ and $x = 2.\overline{3} = 2\frac{1}{3} = \frac{7}{3}$

Tia and Tehani are surprised when they realize that both of these equations have more than one answer.

2. Explain why there is a second solution to each of these problems.

Because when you square root both sides you don't account for the negative solution

3. Modify Tia's and Tehani's algebraic approaches so they will find both solutions.

$9(x-3)^2 = 4$

$\sqrt{9(x-3)^2} = \pm\sqrt{4}$

$\frac{3(x-3)}{3} = \frac{\pm 2}{3}$

$x - 3 = \pm\frac{2}{3}$

$x = \pm\frac{2}{3} + 3 \rightarrow x = \frac{2}{3} + 3$ and $x = -\frac{2}{3} + 3$

$x = \frac{11}{3}$ and $x = \frac{7}{3}$

4. Use Zac's graphing calculator approach to solve the following problem.

Carlos and Clarita deposited \$300 in an account earning 5% interest. They want to take the money out of the account when it has doubled in value. To the nearest month, when can they withdraw their money?

$300(1.05)^x = 600 \rightarrow y_1 = 300(1.05)^x$ $y_2 = 600$

when x= it will have double in value